Centre for Crop and Disease Management

Curtin Biometry and Agricultural Data Analytics

Curtin University

Response-variation subspaces for multivariate varying-coefficient fields

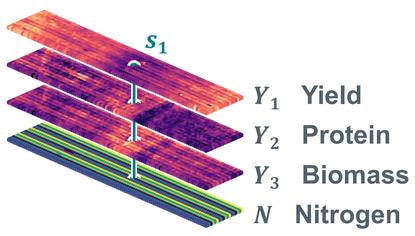
Kai A. Bagley · Julia Easton · Suman Rakshit · Zhanglong Cao · Arnold Salvacion · Mark Gibberd

ABSTRACT

For paddock input decisions, we study shared spatial changes in nitrogen effects on yield, protein and biomass, and establish when the experiment can distinguish them. Constant effects remain unrestricted. This study uses observed yield and simulated protein and biomass, and controlled simulations to test methods to assess estimation and uncertainty. Shared directions can remain identifiable even when individual coefficients are not.

1 Multivariate spatial data

Each location has a nitrogen rate N and three responses Y : yield, protein and biomass. How does the effect of nitrogen on these responses vary across the field, and which changes do they share?

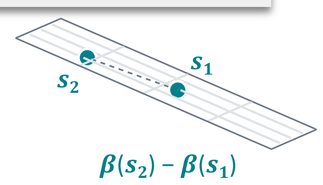


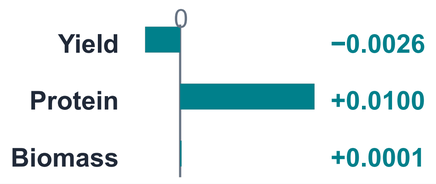
2 Variation subspace

$$\mathcal{W}_{\text{sp}} = \text{span}\{\beta(s_2) - \beta(s_1) : s_1, s_2 \in D\}$$

Let $\boldsymbol{\beta}(s)$ contain the three nitrogen effects at location s in field D, per kg/ha N. We assume a linear response and measure effects in response standard deviations.

Comparing any two locations s_1 and s_2 gives a vector of effect differences. $\boldsymbol{w}_{\text{sp}}$ collects their independent directions across response effects. Constant effects give zero, and add nothing to $\boldsymbol{w}_{\text{sp}}$.





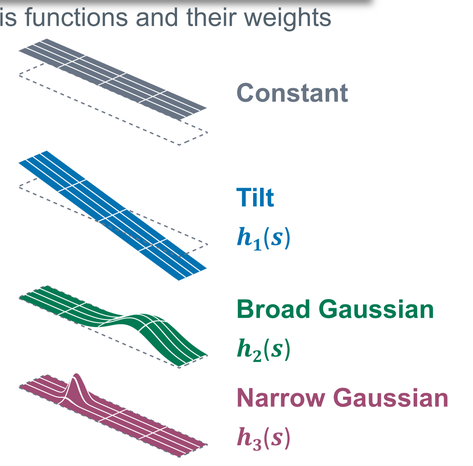
3 Basis functions

$$\beta(s) = \underbrace{b_0}_{\text{constant effects}} + \underbrace{b_1 h_1(s) + b_2 h_2(s) + b_3 h_3(s)}_{\text{chosen basis functions and their weights}}$$

For the simulated example, model varying nitrogen effects with three basis functions: a tilt $\boldsymbol{h_1}$ and Gaussian patches $\boldsymbol{h_2}$, $\boldsymbol{h_3}$.

We then fit weights $\boldsymbol{b_1}$, $\boldsymbol{b_2}$, $\boldsymbol{b_3}$ and constant effects $\boldsymbol{b_0}$ from the data. Each $\boldsymbol{b}$ contains one weight per outcome.

This defines the possible treatment-effect surfaces. The data determine their weights.

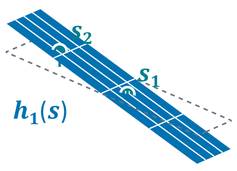


4 Observation model

$$\underbrace{y_i}_{\text{outcomes at } s_i} = \underbrace{m_i}_{\text{background}} + \underbrace{N_i \beta(s_i)}_{\text{N × effect}} + \underbrace{\varepsilon_i}_{\text{noise}}$$

Z is the known nitrogen × basis table, with one row per location and one column per basis function.

X_0 holds known background inputs (e.g. topography and spatial trends), plus a column N for the constant effect b_0.



N Rate x Basis function

	N_i (kg/ha)		$h_1(s_i)$		Z_{i1}
s_1	0.0	×	-0.1042	=	0.00
s_2	131.5	×	0.5417	=	71.23

Z's columns have weights $\boldsymbol{b_1}$, $\boldsymbol{b_2}$ and $\boldsymbol{b_3}$. The table shows two entries from column 1.

5 Identifiability

Identifiability asks whether the trial can distinguish nitrogen-effect patterns from background effects. We check which shared response directions remain distinguishable after accounting for that background.

If every location gets the same nitrogen rate, **does a patch of high yield reflect background fertility or a stronger nitrogen effect?** Applying different nitrogen rates across the field can help separate them.

6 Shared structure

$$B = WA \qquad \underbrace{\text{rank}(B) \le 2}_{\text{at most two shared patterns}}$$

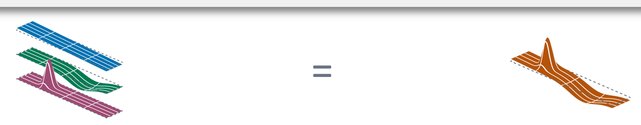
B is a 3 × 3 matrix: one row per outcome, with $\boldsymbol{b_1}$, $\boldsymbol{b_2}$ and $\boldsymbol{b_3}$ as its columns. The weights a below combine the basis functions into shared patterns f_1 and f_2.

The three outcome effects can vary together in at most two independent spatial patterns.

Here we choose two for illustration. For real data, compare predictions on plots or spatial blocks left out of fitting.

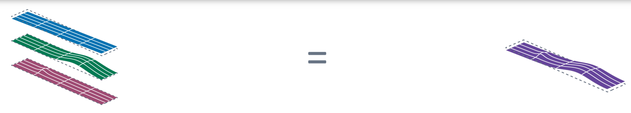
$$a_{11}h_1(s) + a_{12}h_2(s) + a_{13}h_3(s) = f_1(s)$$

$a_{11}h_1(s)$
$a_{12}h_2(s)$
$a_{13}h_3(s)$



$$a_{21}h_1(s) + a_{22}h_2(s) + a_{23}h_3(s) = f_2(s)$$

$a_{21}h_1(s)$
$a_{22}h_2(s)$
$a_{23}h_3(s)$



7 Constrained maximum likelihood

$$\mu = X_0 A_0^\top + ZB^\top \qquad \underbrace{\arg\min_{A_0, B} \|Y - \mu\|^2_{\text{noise}}, \quad \text{rank}(B) \le 2}_{\text{Objective to minimise}}$$

Predict responses from the weights

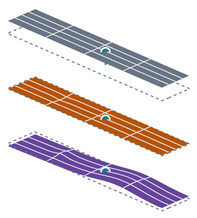
We estimate $\boldsymbol{b_1}$, $\boldsymbol{b_2}$ and $\boldsymbol{b_3}$ in B, together with the background weights A_0, including b_0. The left equation uses these weights and known inputs X_0 and Z to predict all three responses at every location.

We choose the weights that **minimise the objective** on the right. It adds up squared differences between measurements Y and predictions μ, accounting for spatial and response correlation. B must have rank at most two.

With fixed separable Gaussian noise, singular value decomposition (SVD) gives the exact maximum-likelihood solution [2,3].

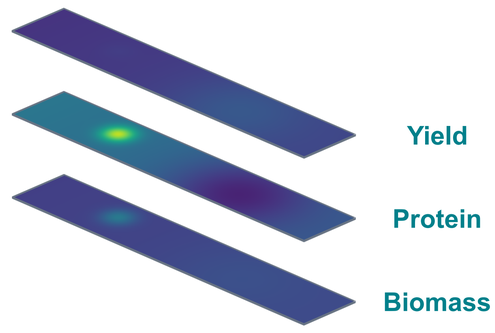
8 Final mix

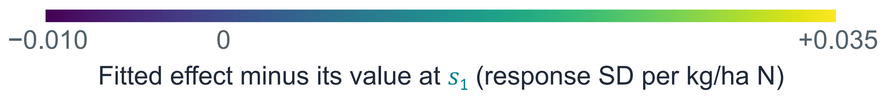
Full yield effect $\beta(s) = b_0 + w_1 f_1(s) + w_2 f_2(s)$



Rewrite fitted B using a and w_1, w_2; each w has one weight per outcome.

Add each outcome's two pattern shares to its constant effect.





9 Simulation checks

True patterns	Our limit	GLS limit	RMSE gain	Coverage
1 pattern	1	3	38%	90%
2, equally strong	2	3	14%	96%
2, second 3% as strong	2	3	5%	88%

400 datasets per row; our fit is given the true rank. GLS (generalised least squares) uses the same basis and spatial noise model. Gain: lower mean RMSE against the true centred effect maps, versus GLS. Coverage: % of our 95% bootstrap regions containing the true subspace.

References

[1] Gelfand et al. (2003). JASA 98, 387–396.
[2] Anderson (1951). Ann. Math. Statist. 22, 327–351.
[3] Eckart & Young (1936). Psychometrika 1, 211–218.
[4] Brunsdon et al. (1996). Geogr. Anal. 28, 281–298.

AAGI Analytics for the Australian Grains Industry

GRDC Grains Research & Development Corporation

Adelaide University

Curtin University

The University Of Queensland Australia

The support of the Grains Research and Development Corporation (GRDC) provided through the Analytics for the Australian Grains Industry (AAGI) project is acknowledged.